**Harder Simultaneous equations**

**1.** Solve $x²-y²=15$ and $x+y=3$ for $x$ and $y$.

 $\left\{\begin{array}{c}x²-y²=15…(1)\\x+y=3…(2)\end{array}\right.$

 Let $x=1.5+k$, by (2) $y=1.5-k$.

 Substitute in (1), $\left(1.5+k\right)²-\left(1.5+k\right)²=15$

 $∴6k=15⟹k=2.5$

 $∴x=1.5+2.5=4, y=1.5-2.5=-1$.

**2.** Solve $\left\{\begin{array}{c}\left(x+1\right)^{2}\left(y+1\right)^{2}=27xy\\\left(x^{2}+1\right)\left(y^{2}+1\right)=10xy\end{array}\right.$

 $\left\{\begin{array}{c}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^{2}\left(\sqrt{y}+\frac{1}{\sqrt{y}}\right)^{2}=27…(1)\\\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)=10…(2)\end{array}\right.$

 Put $u=\sqrt{x}+\frac{1}{\sqrt{x}} , v=\sqrt{y}+\frac{1}{\sqrt{y}}$

 Then $x+\frac{1}{x}=u^{2}-2, y+\frac{1}{y}=v^{2}-2…(3)$.

 The system of equations become

 $\left\{\begin{array}{c}u^{2}v^{2}=27…(4)\\\left(u^{2}-2\right)\left(v^{2}-2\right)=10…(5)\end{array}\right.$

 From (5), $u^{2}v^{2}-2\left(u^{2}+v^{2}\right)+4=10$

 From (4), $27-2\left(u^{2}+v^{2}\right)+4=10$

 $u^{2}+v^{2}=\frac{21}{2}…(6)$

 From (6) and (4), $u^{2},$ $v^{2}$ are roots of

 $t^{2}-\frac{21}{2}t+27=0⟹2t^{2}-21t+54=0⟹t=6,\frac{9}{2}$

 $∴u^{2},$ $v^{2}=6,\frac{9}{2}⟹$ $u^{2}-2,v^{2}-2=4, \frac{5}{2}⟹ x+\frac{1}{x} , y+\frac{1}{y}=4, \frac{5}{2}$

 (a) $x+\frac{1}{x}=4⟹x^{2}-4x+1=0⟹x=2\pm \sqrt{3}$

 (b) $x+\frac{1}{x}=\frac{5}{2}⟹2x^{2}-5x+2=0⟹x=2, \frac{1}{2}$

 Since (1),(2) is symmetric in $x,y$, the solutions are

 $\left(x,y\right)=\left(2, 2\pm \sqrt{3}\right), \left(\frac{1}{2}, 2\pm \sqrt{3}\right), \left(2\pm \sqrt{3}, 2\right), \left(2\pm \sqrt{3}, \frac{1}{2}\right)$.

**3.** Solve the system, $\left\{\begin{array}{c}ax+by=\left(x-y\right)^{2}\\by+cz=\left(y-z\right)^{2}\\cz+ax=\left(z-x\right)^{2}\end{array}\right.$ where $a,b, c>0$?

 $\left\{\begin{array}{c}ax+by=\left(x-y\right)^{2}…(1)\\by+cz=\left(y-z\right)^{2}…(2)\\cz+ax=\left(z-x\right)^{2}…(3)\end{array}\right.$

 First we solve (1) and (2) by rewriting as:

 (i) $ax+by=\left(x-y\right)^{2}⟹y^{2}-y\left(2x+b\right)+\left(x^{2}-ax\right)=0…(1)$

 (ii) $by+cz=\left(y-z\right)^{2}⟹y^{2}-y\left(2z+b\right)+\left(z^{2}-cz\right)=0…(2)$

 $\left(1\right)≡\left(2\right), \left\{\begin{array}{c}2x+b=2z+b…(4)\\x^{2}-ax=z^{2}-cz…(5)\end{array}\right.$

 From (4), $x=z…(6)$

 $\left(6\right)\downright \left(5\right), x^{2}-ax=x^{2}-cx⟹\left(a-c\right)x=0…(7)$

 (i) If $a\ne c, x=z=0$, substitute in (1), $y^{2}-yb=0⟹y=0 or b$

 Note that the equation $cz+ax=\left(z-x\right)^{2}…(3)$ is satisfied for $x=z=0$.

 (ii) If $a=c$, from (6), $x=z=t$ is a free variable.

 Substitute in $\left(3\right), at+at=\left(t-t\right)^{2}⟹t=0$

 $\left(x,y,z\right)=\left(0,0,0\right)$

 Similarly we can solve (2), (3) and substitute in (1) or solve (3), (1) and substitute in (2).

 Complete solution: $\left(x,y,z\right)=\left(0,0,0\right),\left(a,0,0\right),\left(0,b,0\right)\left(0,0,c\right)$**.**

**4.** If $x+3y+5z=200$ and$ x+4y+7z=225$, then what is $x+y+z$ equal to?

 $S:\left\{\begin{array}{c}x+3y+5z=200…(1)\\x+4y+7z=225…(2)\\x+y+z=k …(3)\end{array}\right.$

 The coefficient determinant of $ S$ is zero.

 In order the system to have solution, there must be a free valuable.

 Let this be $z$, and interestingly, $z$ can be any value you like.

 For simplicity, put $z=0$.

 (1) and (2) becomes $\left\{\begin{array}{c}x+3y=200…(4)\\x+4y=225…(5)\end{array}\right.$

 Solving, $x=125, y=25$.

 $x+y+z=125+25+0=150$

**5.** Solve $\left\{\begin{array}{c}x+\sqrt{y}=7\\y+\sqrt{x}=11\end{array}\right.$

 **Method 1**

 $\left\{\begin{array}{c}x+\sqrt{y}=7…(1)\\y+\sqrt{x}=11…(2)\end{array}\right.$

 Put $u=\sqrt{x}, v=\sqrt{y}$. The system of equation becomes

 $\left\{\begin{array}{c}u^{2}+v=7 …(3)\\u+v^{2}=11…(4)\end{array}\right.$

 From (1), $v=7-u^{2}…\left(5\right)$

 $\left(5\right)\downright \left(4\right), u+\left(7-u^{2}\right)^{2}=11$

 $u^{4}-14u^{2}+u+38=0$

 $∴u=2, 3.1313, -3.2832, -1.8481$.

 Since $u=\sqrt{x}>0$

 $∴u=2, 3.1313…(6)$

 $∴x=u^{2}=4, 9.805 $

 (6)$\downright $(5), $v=3,-89.14 $

 Since $v=\sqrt{y}>0, v=3$

 $∴y=v^{2}=9$

 $∴x=4,y=9$ is the only answer.

 **Method 2 Numerical method**

 $x+\sqrt{y}=7⟹x=7-\sqrt{y}$

 $y+\sqrt{x}=11⟹y=11-\sqrt{x}$

 We set up our iterative formula, $\left\{\begin{array}{c}y\_{n}=11-\sqrt{x\_{n}}…(1)\\x\_{n+1}=7-\sqrt{y\_{n}}…(2)\end{array}\right.$

 We choose $x\_{1}=1, $from (1), $y\_{1}=10$.

 $∴\left(x\_{1},y\_{1}\right)=\left(1,10\right)$

 From (2), $x\_{2}=7-\sqrt{10}≈3.837722$

 From (1), $y\_{1}=11-\sqrt{3.837722}≈9.0409895$

 $∴\left(x\_{2},y\_{2}\right)=\left(3.837722, 9.0409895\right)$

 Continue in this way:

 $∴x=4,y=9$

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